MTH 406: Differential geometry of curves and surfaces

Homework III

Problems for practice

1. Determine whether the following subsets of $\mathbb{R}^3$ are regular surfaces. If they are, find coordinate neighborhoods that cover them, otherwise, explain why they are not regular.

   (a) The cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

   (b) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } z = 0\}$.

   (c) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1 \text{ and } z = 0\}$.

   (d) The two-sheeted cone $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$.

   (e) $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in V \text{ and } z = 0\}$, where V is an open subset of the xy-plane.

   (f) $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in C\}$, where C is the lemniscate (i.e. Figure 8 curve) in the xy-plane.

   (g) $\{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$.

2. Find the critical points, critical values, and the regular values for each of the following functions.

   (a) $f(x, y, z) = (x + y + z - 1)^2$.

   (b) $f(x, y, z) = xyz^2$.

   (c) $f(x, y, z) = z^2$.

3. Determine whether the following maps are differentiable.

   (a) The map $d : S \to \mathbb{R}$ given by $d(p) = |p - p_0|$, where S is a regular surface and $p_0 \in \mathbb{R}^3 \setminus S$.

   (b) The orthogonal projection $\pi : S \to \mathbb{R}^2$, where S is a regular surface.

4. Show that the following maps are diffeomorphisms.

   (a) The antipodal map $A : S^2 \to S^2 : (x, y, z) \stackrel{A}{\longmapsto} (-x, -y, -z)$.

   (b) The stereographic projection $\rho : S^2 \setminus \{N\} \to \mathbb{R}^2$, where N is the north pole.

   (c) The rotation of a surface of revolution about its axis of rotation.

   (d) Reflection of a surface about a plane that it is symmetric about.